A GENERALIZED THEORY OF PLASTICITY

POSPLOŠENA TEORIJA PLASTIČNOSTI

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A closed solution of the plane problem of the generalized theory of plasticity and a model of the complex plastic medium were theoretically developed. Solutions with the use of the deformation theory and the theory of plastic yielding were developed. The solution for a simple strengthening medium was deduced.

Key words: metal plasticity, analytical solution, mathematical model, plastic medium, process parameters

Razvitja sta bila zaprta rešitev spošne teorije plastičnosti in teoretičen model kompleksnega plastičnega medija. Razvite so bile rešitve z uporabo teorije deformacije in teorije plastičnega tečenja. Razvita je bila rešitev za preprost utrditveni medij.

Ključne besede: plastičnost kovin, analitična rešitev, matematični model, plastični medij, parametri procesa

1 INTRODUCTION

A characteristic of the new method based on a closed solution of the plane problem of theory of plasticity is a simplified analysis of the deformation mode of the medium and the theoretical connection to the medium mechanical characteristics through the process parameters. The analytical solution of the plane problem of the theory of plasticity for a strengthening medium is known. The developed complex model for the strengthening of the plastic medium is based on the shear resistance to the plastic deformation and is a function of the coordinates of the nucleus of deformation. This approach offers a new possibility to evolve a new solution for a problem, including the generalized theory of plasticity. The approach includes equations and criteria: an equilibrium equation, and the criteria of yielding, the equation of incompressibility, of the deformation rate and the deformation as well as equations of continuity of the deformation rate and the deformation:

- the equilibrium equations
  \[
  \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0; \quad \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0
  \]

- the criterion of yielding
  \[
  (\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2
  \]

- the constraint equations for the rates of deformation and deformation
  \[
  \frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{\xi_x - \xi_y}{\gamma_{xy}} = F_1; \quad \frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{\xi_x - \xi_y}{\gamma_{xy}} = F_2
  \]

- the equations of incompressibility for the rates of deformation and the deformation
  \[
  \hat{\xi}_x + \hat{\xi}_y = 0; \quad \hat{\varepsilon}_x + \hat{\varepsilon}_y = 0
  \]

- the equation of continuity for the deformation rates and the deformation
  \[
  \frac{\partial^2 \xi_x}{\partial y^2} + \frac{\partial^2 \xi_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial y \partial x}
  \]

- the equation of heat conductivity
  \[
  a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{\partial T}{\partial t}
  \]

The model of the complex plastic medium is defined with

\[
T = \chi \left(\frac{H_1}{H_2} \cdot \left(\Gamma_1 \right) \cdot \left(\Gamma_2 \right) \cdot \left(T \right) \right)
\]

The system of equations (1) includes the equations of the deformation theory of plasticity and the theory of plastic yielding with the addition of the equation of heat conductivity. The model (2) is a real strengthening medium with the boundary conditions for stresses

\[
\tau_n = T_1 \cdot \sin \frac{A\Phi - 2\alpha}{2} \cdot T_1 = k
\]

or

\[
\tau_n = \frac{\sigma_x - \sigma_y}{2} \cdot \sin \frac{2\alpha}{2} \cdot \tau_{xy} \cdot \cos 2\alpha
\]

The additional conditions are given by the specific contact forces (3) of the change of friction according to the sinusoidal law of deformational and high-speed strain hardening. All the intensities and the temperature depend on the coordinates of the deformation nucleus.
2 THEORETICAL DEVELOPMENT

With the aim to obtain the model (2), let us consider three second-order equations in form of non-uniform hyperbolic partial derivatives:

\[
\begin{align*}
\frac{\partial^2 \tau_{\sigma}}{\partial x^2} - \frac{\partial^2 \tau_{\sigma}}{\partial y^2} &= 2 \frac{\partial^1}{\partial y \partial x} \left[ k_{\sigma} \tau_{\sigma}^2 \right], \\
\frac{\partial^2 \xi_{x}}{\partial x^2} - \frac{\partial^2 \xi_{x}}{\partial y^2} &= 2 \frac{\partial^1}{\partial y \partial x} F_{1} \xi_{x} \\
\frac{\partial^2 \xi_{y}}{\partial x^2} - \frac{\partial^2 \xi_{y}}{\partial y^2} &= 2 \frac{\partial^1}{\partial y \partial x} F_{2} \xi_{y}
\end{align*}
\tag{4}
\]

The boundary conditions (3) correspond to the substitution \( r_{\sigma} = k \sin \Phi \). A complex dependence of the coordinates is assumed with \( k = f(\Gamma, \Pi, T, x, y) \). In this case, \( k = k_{\sigma} \exp \theta \), with \( \theta = f(\Gamma, \Pi, T, x, y) \), with \( \Gamma, \Pi, T \) standing for the intensity of the deformation, the rates of deformation and the temperature.

The derivatives are taken as for the complex function,\(^a\) and after substitution in the first equation (3) we obtain:

\[
\begin{align*}
&\left\{ \left( \theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{x} \right) \right\} + \left\{ \left( \theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{y} \right) \right\} - \\
&\left\{ \left( \theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{x} \right) - A \Phi_{1} \right\} + 2 A \Phi_{2} \\
&- \sin(\Phi) + 2 \left\{ \left( \theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{x} \right) + A \Phi_{2} \right\} - \\
&- \left\{ \left( \theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{y} \right) - A \Phi_{1} \right\} - 2 \left\{ \left( \theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{y} \right) + A \Phi_{2} \right\}
\end{align*}
\]

Equation (5) is equal to zero if the parts in the square brackets are equal to zero. Then,

\[
\begin{align*}
\theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{x} &= A \Phi_{1} \\
\theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{y} &= A \Phi_{2} \\
\theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{x} &= -A \Phi_{2} \\
\theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{y} &= -A \Phi_{1}
\end{align*}
\]

\[
\begin{align*}
A \Phi_{1} &= -\left( \theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{x} \right), \\
A \Phi_{2} &= \left( \theta_{\mu} H_{\mu} + \theta_{x} \Gamma_{x} + \theta_{y} \Gamma_{y} + \theta_{T} T_{y} \right)
\end{align*}
\]

The operations with the complex function allow us to determine the exponent index as the sum of three functions accounting for the effect of the deformation degree and the rate, and of the temperature:

\[
\theta = -A \Phi + \theta_{x} + \theta_{y} = -\left( \lambda_{x} \theta_{x} + \lambda_{y} \theta_{y} + A \theta \right)
\]

The shear resistance and the components of the tensor of the stresses are:

\[
k = C_{\sigma} \exp(-\lambda_{1} \theta_{x}) \exp(-\lambda_{2} \theta_{y}) \exp(-\lambda_{3} \theta) \sin(\Phi \Phi)
\tag{6}
\]

\[
\begin{align*}
\sigma_{x} &= C_{\sigma} \exp(-\lambda_{1} \theta_{x}) \exp(-\lambda_{2} \theta_{y}) \exp(-\lambda_{3} \theta) \cos(A \Phi \Phi) + \\
&+ \sigma_{y} + f(\Phi \Phi) + C \\
\sigma_{y} &= C_{\sigma} \exp(-\lambda_{1} \theta_{y}) \exp(-\lambda_{2} \theta_{x}) \exp(-\lambda_{3} \theta) \cos(A \Phi \Phi) + \\
&+ \sigma_{x} + f(\Phi \Phi) + C
\end{align*}
\]

with \( \theta_{x} = (\theta_{x})_{1} + (\theta_{x})_{2} + (\theta_{x})_{3} = -A \Phi_{1} \), \( \theta_{y} = (\theta_{y})_{1} + (\theta_{y})_{2} + (\theta_{y})_{3} = A \Phi_{2} \).

By substituting the stress values into the equation of constraint we obtain:

\[
\begin{align*}
&\frac{\sigma_{x} - \sigma_{y}}{2\tau_{\sigma}} = \cot \Phi \Phi, \\
&\frac{\xi_{x} - \xi_{y}}{\gamma_{\sigma}} = \cot B_{1} \Phi \\
&\frac{\varepsilon_{x} - \varepsilon_{y}}{\gamma_{\sigma}} = \cot B_{2} \Phi
\end{align*}
\]

It is possible to establish the relation between the shear and the linear figures of the deformation rates and the deformations. Taking into account the equations of non-compressibility we obtain:

\[
\begin{align*}
\gamma_{\sigma} &= 2 - \frac{1}{F_{1}}, \\
\varepsilon_{x} &= \frac{1}{F_{2}}
\end{align*}
\]

In order to simplify, we define:

\[
\begin{align*}
\xi_{x} &= C_{\epsilon} \exp(\lambda_{1} \theta_{x}) \cos B_{1} \Phi \\
\varepsilon_{x} &= C_{\epsilon} \exp(\lambda_{2} \theta_{y}) \cos B_{2} \Phi
\end{align*}
\]

By substituting these relations into the equations of continuity of the deformation rate and the deformation (1) or (4), we obtain:

\[
\begin{align*}
&\left[ \theta_{x} + (B_{1} \Phi_{x}) + (\theta_{x} - (B_{1} \Phi_{x})) \right] \sin B_{1} \Phi + \\
&2 \left[ (B_{2} \Phi_{x} - \theta_{y}) \left( B_{1} \Phi_{x} + (B_{1} \Phi_{y} - B_{1} \Phi_{x}) \right) \right] \cos B_{2} \Phi
\end{align*}
\]

as well as

\[
\begin{align*}
&\left[ \theta_{y} + (B_{2} \Phi_{y}) + (\theta_{y} - (B_{2} \Phi_{y})) \right] \sin B_{2} \Phi + \\
&2 \left[ (B_{2} \Phi_{y} - \theta_{y}) \left( B_{1} \Phi_{x} + (B_{1} \Phi_{y} - B_{1} \Phi_{x}) \right) \right] \cos B_{2} \Phi
\end{align*}
\]

Brackets identical to (5) appear in equations (7) and (8). For

\[
\begin{align*}
\theta_{x} &= -B_{1} \Phi_{x}, \\
\theta_{y} &= -B_{1} \Phi_{y}, \\
\theta_{x} &= -B_{2} \Phi_{x}, \\
\theta_{y} &= -B_{2} \Phi_{y}
\end{align*}
\]

the equations are transformed into identities, with, \( \theta_{x} = -B_{1} \theta_{x} \), \( \theta_{y} = -B_{2} \theta_{y} \) as the indices of the exponents of the functions determining the fields of the deformation rate and the deformation, \( B_{1} \Phi \) and \( B_{2} \Phi \) are the trigonometric functions determining the fields of the deformation rate and the deformation.

The expressions for the deformation rate and the deformation are:

\[
\begin{align*}
\sigma_{x} &= C_{\sigma} \exp(-\lambda_{1} \theta_{x}) \exp(-\lambda_{2} \theta_{y}) \exp(-\lambda_{3} \theta) \cos(A \Phi \Phi)
\end{align*}
\]
\[ \dot{\theta}_s = -\frac{\xi}{\gamma} = C_{\xi} \cdot \exp \theta_1 \cdot \cos \theta \cdot B \cdot \Phi = C_{\xi} \cdot \exp(-B \cdot \theta) \cdot \cos B \cdot \Phi = \frac{1}{2} \cdot C_{\xi} \cdot \exp(-B \cdot \theta) \cdot \sin B \cdot \Phi \]

\[ \gamma = C_{\xi} \cdot \exp \theta_1 \cdot \sin \theta \cdot B \cdot \Phi = C_{\xi} \cdot \exp(-B \cdot \theta) \cdot \sin B \cdot \Phi \]

\[ H_\gamma = 2C_{\xi} \cdot \exp \theta_1 \cdot \sin B \cdot \Phi \]

\[ \epsilon = -\epsilon_1 = C_{\xi} \cdot \exp \theta_1 \cdot \cos B \cdot \Phi = -C_{\xi} \cdot \exp(-B \cdot \theta) \cdot \cos B \cdot \Phi \]

\[ \gamma = C_{\xi} \cdot \exp \theta_1 \cdot \sin B \cdot \Phi = C_{\xi} \cdot \exp(-B \cdot \theta) \cdot \sin B \cdot \Phi \]

\[ \Gamma = 2C_{\xi} \cdot \exp \theta_1 \cdot \sin B \cdot \Phi \]

\[ \left( \theta_1 \right)_x = B \cdot \Phi \]

\[ \left( \theta_1 \right)_y = -B \cdot \Phi \]

With a comparison of expressions (9), (10) and (7) we can confirm that all the expressions have functional dependencies on the coordinates \( \theta \) and \( \Phi \) (the indices of the exponents and the examples of the trigonometric functions).

It is of interest to obtain similar dependencies for the solution of the temperature fields that could allow us to solve the problem theoretically. Let us consider a differential equation for the stationary temperature field

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \]

For this case we look for the solution in the form of

\[ T = C \cdot \exp \left( \theta \right) \cdot \cos \left( B \cdot \Phi \right) \]

with: \( \left( \theta \right)_x = -B \cdot \Phi \)

We will demonstrate that expression (11) is a solution of the Laplace equation. By substituting the derivatives (11) into the equation of the heat conductivity and a simplification we obtain:

\[ \left( \theta \right)_x + \left[ B \cdot \Phi \right] \left( \left( \theta \right)_y \right) + \left( \left( \theta \right)_y \right) + \left( \theta \cdot B \cdot \Phi \right) + \left( \left( \theta \right)_y \right) + \left( \left( \theta \right)_y \right) + \left( \theta \cdot B \cdot \Phi \right) + \left( \left( \theta \right)_y \right) + \left( \theta \cdot B \cdot \Phi \right) + \left( \left( \theta \right)_y \right) + \left( \theta \cdot B \cdot \Phi \right) = 0 \]

In the case of equality to zero, the brackets \( \left[ \left( \theta \right)_x + \left( \theta \right)_y + \left( \theta \cdot B \cdot \Phi \right) \right] \) in equation (12) establish a relation in the form of:

\[ \left( \theta \right)_x = -B \cdot \Phi \]

The last correspond to Cauchy-Riemann conditions and are functions determined by the Laplace equation corresponding to relation (11).

From the comparison of the solutions (7) to (11) (conditions superimposed on functions) it was concluded that \( \theta_1 = -B \theta \) for the stressed and deformed conditions and the temperature fields can be used to determine a common parametric function, which is included into the fields of the stresses, the deformations, the rates of deformation and the temperatures, allowing us to express them mathematically, one with another. Thus,
By substituting the components of the tensor of stresses into (14), we obtain

\[\sigma_x = -k_y \frac{\exp(\theta - \theta_0)}{\cos A\Phi} \cos A\Phi + k_0\]
\[\sigma_y = -3k_y \frac{\exp(\theta - \theta_0)}{\cos A\Phi} \cos A\Phi + k_0\]
\[\tau_{xy} = k_0 \frac{\exp(\theta - \theta_0)}{\cos A\Phi} \sin A\Phi\]  

(15)

The results of the calculation according to equations (15) in Figures 1 to 4 show that the distribution of the contact stresses is related to the factor of the shape of the stress nucleus \(h/\delta\) and the friction coefficient \(f\), the relative normal stresses \(\sigma_x/2k_0\) and the relative tangent stresses \(\tau_{xy}/k_0\). The results of the calculation correspond to the real distribution diagrams of the contact stresses. It should be emphasized that expressions (15) are uniform for the entire nucleus of deformation and there is no need to break it into separate zones of contact friction. Figures 5 and 6 show the distribution of stresses...
in the nucleus of deformation, which also include the
shape factor and the friction coefficient.

The obtained results display qualitatively and quantitatively the general patterns of the distribution fields of
the tensor of stresses over the entire nucleus of de-
formation. The results meet fully the requirements of the
boundary conditions. In particular, the proposed pro-
dure and expressions can be recommended for
the calculation of various problems in applications.

The proposed complex model of a plastic medium
based on the closed solution can be considered as a
generalization of the theory of plasticity, uniting the
theories of deformation and of plastic yielding.

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LIST OF SYMBOLS

\( \sigma \) - normal components of the stress tensor;
\( \tau \) - tangential components of the stress tensor;
\( \xi \) - linear components of the strain-rate tensor;
\( \gamma \) - shear components of the strain-rate tensor;
\( \varepsilon \) - linear deformation along axes x and y;
\( \zeta \) - factor of correspondence between the tangent stress and the
temperature-deformation parameter of the centre to deformation;
\( \Gamma \) - intensity of the shift of the deformation;
\( t_n \) - tangential contact stress on an arbitrary inclined area;
\( \alpha \) - angle of inclination of the contact area;
\( k \) - shearing plastic deformation strength;
\( \Phi \) - harmonic function depending on the coordinates of the deforma-
tion zone and the argument of a trigonometric function;
\( \Phi_0 \) - argument of trigonometric function for \( x = \alpha/2 \) and \( y = \beta/2 \);
\( A \) - constant characterizing the trigonometric function for the state
of stress of the plastic medium;
\( \lambda, \mu \) - constant factor, characterizing the shearing tangent stress in
the zone of reduction
\( \Phi_0, \Phi_1 \) - values taking into account the influence of the factor of fric-
tion;
\( B \) - constant value characterizing the trigonometric function for the
state of strain of the plastic medium;
\( \theta_0 \) - factor exponents for \( x = \alpha/2 \) and \( y = \beta/2 \);
\( \theta \) - harmonic function, exponential index, characterizing the shearing
stress distribution in the zone of reduction;
\( \phi \) - a harmonic function, exponential index, characterizing the distribu-
tion of the rate of shearing in the zone of reduction;
\( C_0 \) - constant value determining the state of stress of the plastic me-
dium;
\( C_1 \) - constant value characterizing the state of strain of the plastic me-
dium;
\( T \) - temperature in the k-th point;
\( T_1 \) - intensity of tangential stress;
\( H \) - intensity of shearing rates;
\( C_T \) - constant value characterizing the temperature field;
\( a \) - coefficient of temperature conductivity;
\( l \) and \( h \) - length and height of the deformation nucleus during strip up-
serting;
\( f \) - friction coefficient;
\( k_0 \) - contact shear resistance at the beginning of the deformation nu-
cleus