BISINGULAR INTEGRAL EQUATIONS WITH THE CONSTANT COEFFICIENTS IN GENERALIZED HÖLDER SPACES

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In the work Bisingular integral equations with the constant coefficients in generalized Hölder spaces are considered. The collocations and reduction methods are improved. It is shown that it suffices to find the coefficients of interpolation Lagrange polynomial (Fourier coefficients respectively) of the right side for construction of approximate solutions. It is established that in the considered case rapidity of convergence of approximate solutions to the exact one depends only upon constructive properties of the right side.

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Н. В. Снижко *Бисингулярные интегральные уравнения с постоянными коэффициентами в обобщённых пространствах Гёльдера*

В работе рассматриваются бисингулярные интегральные уравнения с постоянными коэффициентами в обобщённых пространствах Гёльдера. Обоснованы методы коллокаций и редукции. Показано, что для построения приближённых решений достаточно найти коэффициенты интерполяционного полинома Лагранжа (коэффициенты Фурье соответственно) правой части. Установлено, что в рассматриваемом случае скорость сходимости приближённых решений к точному зависит только от конструктивных свойств правой части.

In the papers [1, 2] the reduction and collocations methods of approximate resolving of bisingular integral equations (BSIE) in generalized Hölder spaces (GHS) were considered. This paper deals with the case, when the solution is obtained in an elementary manner, namely, the characteristic BSIE with the constant coefficients are considered:

$$(K_0\varphi \equiv) a_0\varphi(t,\tau) + a_1(S_1\varphi)(t,\tau) + a_2(S_2\varphi)(t,\tau) + a_{12}(S_{12}\varphi)(t,\tau) = f(t,\tau), \qquad (1)$$

where

$$(S_1\varphi)(t,\tau) = \frac{1}{\pi i}\oint\limits_{\gamma_1} \frac{\varphi(t_0,\tau)}{t_0 - t}dt_0, (S_2\varphi)(t,\tau) = \frac{1}{\pi i}\oint\limits_{\gamma_2} \frac{\varphi(t,\tau_0)}{\tau_0 - \tau}d\tau_0,$$

$$(S_{12}\varphi)(t,\tau) = -\frac{1}{\pi^2}\oint\limits_{\gamma_1}\oint\limits_{\gamma_2}\frac{\varphi(t_0,\tau_0)}{(t_0-t)(\tau_0-\tau)}dt_0 d\tau_0,$$

$f(t,\tau) \in H_\omega(\gamma)$, $\gamma = \gamma_1 \times \gamma_2$ is closed Lyapunov sceleton, $H_\omega(\gamma)$ is GHS with the standard norm [3, 4]:

$$H_\omega(\gamma) \equiv \{f(t,\tau) : \omega(\delta_1,\delta_2;f) \le C_1\omega(\delta_1,\delta_2), \omega_{1,1}(\delta_1,\delta_2;f) \le C_2\Omega_1(\delta_1)\Omega_2(\delta_2)\},$$

$$\|f(t,\tau)\|_{H_\omega} = \|f(t,\tau)\|_C + H(f;\omega) + H^{t\tau}(f;\omega) =$$

$$= \max_{(t,\tau)\in\gamma}|f(t,\tau)| + \sup_{\delta_1^2+\delta_1^2\ne 0}\frac{\omega(\delta_1,\delta_2;f)}{\omega(\delta_1,\delta_2)} + \sup_{\delta_1^2+\delta_1^2\ne 0}\frac{\omega_{1,1}(\delta_1,\delta_2;f)}{\Omega_1(\delta_1)\Omega_2(\delta_2)}.$$

Here $\omega(\delta_1,\delta_2;f)$ is the modulo of continuity of $f(t,\tau)$, $\omega_{1,1}(\delta_1,\delta_2;f)$ is mixed second-order modulo of continuity of $f(t,\tau)$; $\omega(\delta_1,\delta_2)$ is fixed modulo of continuity, $\Omega_1(\delta)$ and $\Omega_2(\delta)$ are the simple one-dimensional moduli of continuity which are assigned to $\omega(\delta_1,\delta_2)$ (so, $\omega(\delta_1,\delta_2)$ is the structural characteristic of the space).

We say $f(t,\tau) \in H_\omega^{(r,s)}(\gamma)$ when r-th and s-th particular derivatives of $f(t,\tau)$ by variables t and τ respectively are belong to $H_\omega(\gamma)$.

We denote

$$a_0 \pm a_1 \pm a_2 + a_3 = \{ {\Delta^0 \atop \Delta_{12}},$$

$$a_0 \mp a_1 \pm a_2 - a_3 = \{ {\Delta^0 \atop \Delta_{12}}.$$

Suppose $\Delta = \Delta^0\Delta^1\Delta^1\Delta^{12} = (a_0^2 - a_1^2 - a_2^2 - a_{12}^2)^2 - 4(a_0a_{12} - a_1a_2)^2 \ne 0$, i.e. the equation normally solvable (Noether's one). It may be noticed that if Δ^0, Δ^1, Δ^2, Δ^{12} are the constants then the equation (1) has a single solution.

According to the collocations method we shall seek an approximate solution of (1) in the form:

$$\varphi_{mn}(t,\tau) = \sum_{k=-m}^{m}\sum_{l=-n}^{n} c_{kl}t^k\tau^l. \tag{2}$$

Unknown constants c_{kl} are determined from the system:

$$\Delta^0\sum_{k=0}^{m}\sum_{l=0}^{n} c_{kl}t_p^k\tau_q^l - \Delta^1\sum_{k=0}^{m}\sum_{l=-n}^{-1} c_{kl}t_p^k\tau_q^l - \Delta^2\sum_{k=-m}^{-1}\sum_{l=0}^{n} c_{kl}t_p^k\tau_q^l +$$

$$+\Delta^{12}\sum_{k=-m}^{-1}\sum_{l=-n}^{-1}c_{kl}t_p^k\tau_q^l=f(t_p,\tau_q),p=0...2m,q=0...2n, \qquad (3)$$

where $\{t_p\}_{p=0}^{2m}$, $\{\tau_q\}_{q=0}^{2n}$ is the system Fejér points on γ. The following statement is proved.

Theorem 1. [1] Let $f(t,\tau)\in H_{\omega^{(1)}}$, $\Delta\neq 0$; $\omega^{(2)}(\delta_1,\delta_2)$ is that $H_{\omega^{(1)}}\subset H_{\omega^{(2)}}$, $\Omega_1^{(1)}(\delta)/\Omega_1^{(2)}(\delta)$, $\Omega_2^{(1)}(\delta)/\Omega_2^{(2)}(\delta)$ are increasing functions and $\ln\delta_1\ln\delta_1\frac{\omega^{(1)}(\delta_1,\delta_2)}{\Omega_1^{(2)}(\delta_1)\Omega_2^{(2)}(\delta_2)}\to 0$, as $\delta_1,\delta_1\to 0$. Then the system (3) is uniquely solvable with any $c_1<m/n<c_2$, and approximate solutions (2) converge in the space $H_{\omega^{(2)}}$ to exact solution $\varphi(t,\tau)$ of (1) with the rapidity

$$\|\varphi-\varphi_{mn}\|_{H_{\omega^{(2)}}}=O\left(\ln m\ln n\frac{\omega^{(1)}(1/m,1/n)}{\Omega_1^{(2)}(1/m)\Omega_2^{(2)}(1/n)}\right).$$

The immediate construction of solutions of the system (3) is rather tedious. It may be simplified. Indeed, let $(L_{mn}f)(t,\tau)=\sum\limits_{k=-m}^{m}\sum\limits_{l=-n}^{n}f_{kl}t^k\tau^l$ is the interpolation Lagrange polynomial of $f(t,\tau)$ by the Fejér points system, $f_{kl}=f(t_k,\tau_l)$. Denote:

$$(L_{mn}f)^{++}(t,\tau)=\sum_{k=0}^{m}\sum_{l=0}^{n}f_{kl}t^k\tau^l,(L_{mn}f)^{+-}(t,\tau)=-\sum_{k=0}^{m}\sum_{l=-n}^{-1}f_{kl}t^k\tau^l,$$

$$(L_{mn}f)^{-+}(t,\tau)=-\sum_{k=-m}^{0}\sum_{l=0}^{n}f_{kl}t^k\tau^l,(L_{mn}f)^{--}(t,\tau)=\sum_{k=-m}^{-1}\sum_{l=-n}^{-1}f_{kl}t^k\tau^l.$$

Using factorization of the operator K_0 it is easy to receive that the system (3) may be written in the form:

$$\begin{cases}\Delta^0c_{kl}=f(t_k,\tau_l),k=0...m,l=0...n\\ \Delta^1c_{kl}=f(t_k,\tau_l),k=0...m,l=-n...-1\\ \Delta^2c_{kl}=f(t_k,\tau_l),k=-m...-1,l=0...n\\ \Delta^{12}c_{kl}=f(t_k,\tau_l),k=-m...-1,l=-n...-1.\end{cases}$$

From this we obtain the solution of the system (3) explicitly:

$$\begin{cases}c_{kl}^*=\frac{f_{kl}}{\Delta^0},k=0...m,l=0...n\\ c_{kl}^*=\frac{f_{kl}}{\Delta^1},k=0...m,l=-n...-1\\ c_{kl}^*=\frac{f_{kl}}{\Delta^2},k=-m...-1,l=0...n\\ c_{kl}^*=\frac{f_{kl}}{\Delta^{12}},k=-m...-1,l=-n...-1.\end{cases}$$

It follows from these formulas that for construction of approximate solutions of characteristic BSIE with the constant coefficients it suffices to find the coefficients of interpolation Lagrange polynomial of the right side.

In the case when γ is the unit bicircle, according to the reduction method we shall seek an approximate solutions of (1) in the form (2). Unknown constants c_{kl} are determined from the system:

$$\begin{cases} \Delta^0 c_{kl} = f_{kl}, k = 0...m, l = 0...n \\ \Delta^1 c_{kl} = f_{kl}, k = 0...m, l = -n... - 1 \\ \Delta^2 c_{kl} = f_{kl}, k = -m... - 1, l = 0...n \\ \Delta^{12} c_{kl} = f_{kl}, k = -m... - 1, l = -n... - 1. \end{cases} \tag{4}$$

where f_{kl} are coefficients of the Fourier series of the function $f(t,\tau)$ by the function system $\{t^k\tau^l\}$:

$$f_{kl} = \frac{1}{4\pi^2}\oint_{\gamma_1}\oint_{\gamma_2}\frac{f(t,\tau)}{t^{k+1}\tau^{k+1}}dtd\tau, k, l = 0, \pm 1, \pm 2...$$

From this we obtain the solution of the system (4) explicitly:

$$\begin{cases} c^*_{kl} = \frac{f_{kl}}{\Delta^0}, k = 0...m, l = 0...n \\ c^*_{kl} = \frac{f_{kl}}{\Delta^1}, k = 0...m, l = -n... - 1 \\ c^*_{kl} = \frac{f_{kl}}{\Delta^2}, k = -m... - 1, l = 0...n \\ c^*_{kl} = \frac{f_{kl}}{\Delta^{12}}, k = -m... - 1, l = -n... - 1. \end{cases}$$

So, the problem of construction of approximate solutions of Noether characteristic BSIE with constant coefficients reduces to finding the Fourier coefficients of the right side. In the case of reduction method the analogue of the theorem 1 is proved also.

Theorem 2. [2] Let $f(t,\tau) \in H_{\omega^{(1)}}$, $\Delta \neq 0$; $\omega^{(2)}(\delta_1,\delta_2)$ is that $H_{\omega^{(1)}} \subset H_{\omega^{(2)}}$, $\Omega_1^{(1)}(\delta)/\Omega_1^{(2)}(\delta)$, $\Omega_2^{(1)}(\delta)/\Omega_2^{(2)}(\delta)$ are increasing functions and $\ln\delta_1 \ln\delta_1 \frac{\omega^{(1)}(\delta_1,\delta_2)}{\Omega_1^{(2)}(\delta_1)\Omega_2^{(2)}(\delta_2)} \to 0$, as $\delta_1, \delta_1 \to 0$. Then the system (5) is uniquely solvable with any $c_1 < m/n < c_2$, and approximate solutions (2) converge in the space $H_{\omega^{(2)}}$ to exact solution $\varphi(t,\tau)$ of (1) with the rapidity

$$\|\varphi - \varphi_{mn}\|_{H_{\omega^{(2)}}} = O\left(\ln m \ln n \frac{\omega^{(1)}(1/m, 1/n)}{\Omega_1^{(2)}(1/m)\Omega_2^{(2)}(1/n)}\right).$$

By the immediate verification it is easy to make sure that in the considered case (Noether characteristic BSIE with constant coefficients) rapidity of convergence of approximate solutions (2) to the exact one depends only upon constructive properties of $f(t,\tau)$. And if we require the particular derivatives of $f(t,\tau)$ are according with certain conditions that estimates from the theorem 1 and theorem 2 may be strengthened. In particular, if $f \in H^{(r,s)}_{\omega^{(1)}}(\gamma)$ [3] that estimations of the convergence rapidity may be improved considerably, namely:

$$\|\varphi - \varphi_{mn}\|_{H_{\omega^{(2)}}} = O\left(\ln m \ln n \left(\frac{1}{m^r} + \frac{1}{n^s}\right) \frac{\omega^{(1)}(1/m, 1/n)}{\Omega_1^{(2)}(1/m)\Omega_2^{(2)}(1/n)}\right).$$

Література

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